Computational Complexity of Lyapunov Stability Analysis Problem for a Class of Nonlinear Systems

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ABSTRACT

Nonlinear control systems can be stabilized by constructing control Lyapunov functions and computing the regions of state space over which such functions decrease along trajectories of the closed-loop system under an appropriate control law. This paper analyzes the computational complexity of these procedures for two classes of control Lyapunov functions. The systems considered are those that are nonlinear in only a few state variables and that may be affected by control constraints and bounded disturbances. This paper extends previous work by the authors, which develops a procedure for stability analysis for these systems with computational complexity that is exponential only in the dimension of the "nonlinear" states and polynomial in the dimension of the remaining states. The main results are illustrated by a numerical example for the case of purely quadratic control Lyapunov functions.

1. INTRODUCTION

Many dynamical systems can be represented by ordinary differential equations in the physical states of the system influenced by other parameters, such as disturbance and control inputs. The focus of state-feedback control theory is to design a control law (a function that maps measured states of the system to control inputs), which produces a desired performance for the system. Very different theories apply to this problem depending on whether the state derivatives are linear or nonlinear functions of the states in the differential equations defining the system. Many simple, straightforward techniques for robust optimal control of linear systems have been developed. Extensions of these methods to nonlinear systems are sometimes possible, but the analogous procedures that result from this exercise cannot typically be executed in a computationally tractable way. As a result, control of nonlinear systems has been a topic of intense research for some time.

Progress on the nonlinear control problem is difficult because of the inherent complexity of methods that are general enough to apply to arbitrary nonlinear systems. One method that has recently come into favor is to construct a stabilizing control law based on a known control Lyapunov function (CLF) for the system (Refs. [2], [13], [28], [30]). A function is a CLF if a control law exists to render it a Lyapunov function for the closed-loop system. The computation of a stabilizing control law is straightforward from any of a number of universal formulas (Refs. [13], [17], [28]) based on the CLF and the system dynamics, so the control synthesis problem is reduced to constructing a CLF for the system and computing the region of state space over which a control exists to stabilize the system based on the given CLF. Recently, the authors have developed a computationally efficient procedure for solving a version of the nonlinear control problem (Problem 1.3 below) based on a given CLF (Refs. [18], [19], [20]). In its general form, the procedure requires one to construct a CLF and to determine the region of state space over which that CLF guarantees stability of the closed-loop system. In this paper, we analyze the computational complexity of these two problems for two important classes of Lyapunov functions.

We consider the problem of designing a control law that stabilizes a nonlinear system in the sense prescribed by Definition 1.2 below. The formal stability definition is a version of the concept of uniform asymptotic stability used in Ref. [16]. That definition is repeated below.

Definition 1.1 (Ref. [16])

Given a system $\dot{x} = f(x,w)$ with $w(t) \in \mathcal{W} \subseteq R^l$ for all $t \geq 0$ and a compact subset $\Omega \subset R^n$, define $\|x\|_\Omega \doteq \inf\{\|x-y\|, y \in \Omega\}$ Then the system is robustly uniformly asymptotically stable with respect to Ω, or RUAS-Ω, if the following conditions hold:

1. For every $x(0) \in R^n$ and $w(t) \in \mathcal{W}$, the solution $x(t)$ is defined for all $t \geq 0$.
2. Uniform stability: There exists a radially unbounded, continuous, strictly increasing function $\delta(\varepsilon)$, with $\delta(0) = 0$, such that, for any $\varepsilon > 0$, $\|x(0)\|_\Omega \leq \delta(\varepsilon)$, $t \geq 0$, and $w(t) \in \mathcal{W}$, we have $\|x(t)\|_\Omega < \varepsilon$.
3. Uniform attraction: For any $r, \varepsilon > 0$, there exists $T > 0$, such that for every $w(t) \in \mathcal{W}$, $\|x(t)\|_\Omega < \varepsilon$, whenever $\|x(0)\|_\Omega < r$ and $t \geq T$.

Definition 1.2

Given a system $\dot{x} = f(x,w)$ with $w(t) \in \mathcal{W} \subseteq R^l$ for all $t \geq 0$, a positively invariant set $\mathcal{X} \subseteq R^n$ and a compact subset $\Omega \subset \mathcal{X}$, the system is robustly uniformly asymptotically stable over $\mathcal{X}$ with respect to Ω, or RUAS($\mathcal{X}$,Ω), if it is RUAS-Ω whenever

$x(0) \in \mathcal{X}$. We call the set $\mathcal{X}$ a region of stability for the system. In particular, we consider the following control synthesis problem.

Problem 1.3

Consider a continuous time, time-invariant, nonlinear system influenced by a control u(t) in a closed subset $\mathcal{U} \subseteq R^m$ and a disturbance $w(t) \in \mathcal{W} \subseteq R^l$. The state vector $x(t) \in R^n$ is partitioned into $x_N \in R^k$ and $x_L \in R^{n-k}$, and the system has the form

$$\begin{bmatrix} \dot{x}_N \\ \dot{x}_L \end{bmatrix} = \begin{bmatrix} f_N(x_N) \\ f_L(x_N) \end{bmatrix} + \begin{bmatrix} A_N(x_N) \\ A_L(x_N) \end{bmatrix} x_L + \begin{bmatrix} g_w^N(x_N) \\ g_w^L(x_N) \end{bmatrix} w + \begin{bmatrix} g_u^N(x_N) \\ g_u^L(x_N) \end{bmatrix} u \tag{1.1}$$

where all functions of x_N are C^1. Construct sets $\Omega \subset \mathcal{X} \subseteq R^n$ containing an equilibrium point at x=0 and a static state-feedback control law $\mu : \mathcal{X} \to \mathcal{U}$ such that the closed-loop system with $u = \mu(x)$ is RUAS($\mathcal{X}$, Ω).

Note that Problem 1.3 includes the problem of analyzing robust stability for an autonomous system without control, since this is just the case $\mathcal{U} = \{0\}$.

Obviously, we would like $\mathcal{X}$ to be as large as possible and Ω as small as possible. When $\mathcal{W} = \{0\}$ and the system is locally asymptotically stabilizable, local stabilization theory yields a set $\mathcal{X}$ such that the system is RUAS($\mathcal{X}$, {0}) (Refs. [31], [13]). To compute the region of stability generally requires computation times that are exponential in the state dimension n; for the system (1.1), however, the computations required to find $\mathcal{X}$ are tractable.

In many applications, the engineer knows from the physics of the problem that only a few physical quantities affect the system dynamics in a nonlinear way, so that the system can be modeled in the form (1.1). Such systems are also considered in Ref. [3], where the output-feedback stabilization problem (with output x_N) is solved based on an output control Lyapunov function, assuming that the solution to the state-feedback stabilization problem is already available and the output CLF can be constructed.

We now define some relevant terms pertaining to a system of the general form given below, where $f \in C^1$ ($R^n \to R^n$) and $g_w(x)$ and $g_u(x)$ are continuous.

$$\dot{x} = f(x) + g_w(x)w + g_u(x)u \tag{1.2}$$

Definition 1.4

A level set of a proper, positive-definite function $V : R^n \to R$ is defined by real numbers $c_2 > c_1 \geq 0$ via $V^{-1}[c_1,c_2] \doteq \{x \in R^n \mid c_1 \leq V(x) \leq c_2\}$.

Definition 1.5 (Ref. [5])

Given a locally Lipschitz function $V : R^n \to R$ and a continuous vector field f on R^n, the Lie derivative of V(x) along f(x) is defined by

$$L_f V(x) \doteq \limsup_{t \to 0+} \frac{V(x + tf(x)) - V(x)}{t}$$

If V(x) is differentiable at a point $x \in R^n$, then $L_f V(x) = \frac{\partial V}{\partial x}(x) f(x)$.

Definition 1.6 (Refs. [12], [28])

Consider a subset $\mathcal{W} \subseteq R^l$, a closed subset $\mathcal{U} \subseteq R^m$, a positive-definite function W(x), and real numbers $c_2 > c_1 \geq 0$. A proper, positive-definite C^1 function V(x) is a robust control Lyapunov function (RCLF) with stability margin W(x) with controls in $\mathcal{U}$ over $V^{-1}[c_1, c_2]$ for the system (1.2) if there exists a control law $\mu : R^n \to \mathcal{U}$ such that

$$\sup_{x \in V^{-1}[c_1,c_2]} \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w + L_{g_u} V(x) \mu(x) + W(x) \leq 0$$

Equivalently,

$$\sup_{x \in V^{-1}[c_1,c_2]} \inf_{u \in \mathcal{U}} \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w + L_{g_u} V(x) u + W(x) \leq 0$$

Remark 1.7

If $\mathcal{U} = R^m$, the condition in Definition 1.6 is equivalent to

$$\sup_{x \in V^{-1}[c_1,c_2] \cap \ker(L_{g_u} V)} \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w + W(x) \leq 0$$

By Definition 1.6, if V(x) is an RCLF, then a static state-feedback control law exists such that the closed-loop system is robustly stable in the sense defined by Definition 1.2. We now proceed to show how to determine whether a given function V(x) is an RCLF over a given level set in Section 2. In Section 3, we investigate the problem of finding the RCLF V(x) that maximizes the volume of the region over which stability can be guaranteed. We also analyze the computational complexity of these procedures in each of these sections. A numerical example is presented in Section 4 to illustrate the findings of the paper. The main results are summarized in Section 5.

2. STABILITY ANALYSIS USING A GIVEN RCLF

Given an RCLF V(x) and a stability margin W(x), our objective is to determine whether V(x) is an RCLF with stability margin W(x) with controls in $\mathcal{U}$ over some level set given by $V^{-1}[c_1, c_2]$ for the nonlinear system (1.1). By Definition 1.6, this stability condition holds if and only if

$$\sup_{x \in V^{-1}[c_1,c_2]} \inf_{u \in \mathcal{U}} \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w + L_{g_u} V(x) u + W(x) \leq 0 \tag{2.1}$$

Since the term $\inf_{u \in \mathcal{U}} L_{g_u} V(x)u$ is a constant over sets of the form $\{x \in R^n \mid L_{g_u} V(x) = \psi^T\}$ for $\psi \in R^m$, it is useful to parameterize sets of this form when evaluating the condition (2.1).

Under certain assumptions that are made precise in Sections 2.1-2.2, the set $\{x \in R^n \mid L_{g_u} V(x) = \psi^T\}$ can be parameterized by x_N and a parameter $\lambda \in R^{n-k-m}$. With this parameterization, we can express V(x) restricted to $\{x \in R^n \mid L_{g_u} V(x) = \psi^T\}$ by a function $V(x_N, \psi, \lambda)$. This parameterization can be chosen so that $V(x_N, \psi, \lambda)$ is minimized at $\lambda = 0$ for each (x_N, ψ).

To analyze stability over the level set $V^{-1}[c_1, c_2]$, we first make the following definitions.

$$\mathcal{Y}(c_2) \doteq \{(x_N,\psi) \in R^k \times R^m \mid V(x_N,\psi,0) \leq c_2\}$$

$$\mathcal{L}(c_1,c_2,x_N,\psi) \doteq \{\lambda \in R^{n-k-m} \mid c_1 \leq V(x_N,\psi,\lambda) \leq c_2\}$$

$$\Gamma(x_N,\psi,\lambda) \doteq \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x)w + [\inf_{u \in \mathcal{U}} \psi^T u] + W(x)$$

$$\Gamma(c_1,c_2,x_N,\psi) \doteq \max_{\lambda \in \mathcal{L}(c_1,c_2,x_N,\psi)} \Gamma(x_N,\psi,\lambda)$$

The set $\mathcal{Y}(c_2)$ is simply the allowable range of (x_N, ψ) in the given level set and $\mathcal{L}(c_1,c_2,x_N,\psi)$ maps the level set to the parameter space of the variable λ.

Proposition 2.1

V(x) is an RCLF with stability margin W(x) with controls in $\mathcal{U}$ over $V^{-1}[c_1, c_2]$ if and only if $\Gamma(c_1,c_2,x_N,\psi) \leq 0$ for all $(x_N, \psi) \in \mathcal{Y}(c_2)$.

The condition in Proposition 2.1 implies that $\Gamma(c_1,c_2,x_N,\psi) \leq 0$ for all $(x_N, \psi) \in R^k \times R^m$. Since $\mathcal{Y}(c_2)$ is compact, we check the condition in Proposition 2.1 by gridding $\mathcal{Y}(c_2)$ and solving a feasibility problem to determine whether $\Gamma(c_1,c_2,x_N,\psi) \leq 0$ for all (x_N, ψ) in the grid.

When $\mathcal{U} = R^m$, we see from Remark 1.7 that we only need to check whether $\Gamma(c_1,c_2,x_N,\psi) \leq 0$ for $\psi = 0$. In this case, it is only necessary to grid over the permissible values of x_N, which is a compact region of dimension k rather than dimension (k + m). Therefore, Proposition 2.1 holds with the following modified definition.

$$\mathcal{Y}(c_2) \doteq \{(x_N,\psi) \in R^k \times R^m \mid \psi = 0, V(x_N,0,0) \leq c_2\}$$

Remark 2.2

The procedure just described also applies in the analysis of closed-loop robust stability under a control law of the form $\mu(x) = \mu_N(x_N) + \mu_L(x_N)\, x_L$. In this case, the control is already fixed, so we do not need to parameterize sets of the form $\{x \in R^n \mid L_{g_u} V(x) = \psi^T\}$. In other words, we simply parameterize the level set $V^{-1}[c_1, c_2]$ by the variable x_N. The only additional step is to check that $\mu(x) \in \mathcal{U}$ over $V^{-1}[c_1, c_2]$, but this is straightforward for certain common classes of control constraint sets (Ref. [22]).

In Sections 2.1-2.2, we fill in the details of this stability analysis procedure for two specific Lyapunov function classes and evaluate the computational complexity of the procedure in each case.

2.1 Quadratic RCLF with Constant P Matrix

In this section, we fill in the details of the stability analysis procedure for the special case of a standard quadratic RCLF and stability margin of the form

$$V(x) = x^T P x = x_N^T P_{NN} x_N + x_N^T P_{NL} x_L + x_L^T P_{LN} x_N + x_L^T P_{LL} x_L \quad (2.2)$$

$$W(x) = x^T Q x = x_N^T Q_{NN} x_N + x_N^T Q_{NL} x_L + x_L^T Q_{LN} x_N + x_L^T Q_{LL} x_L \quad (2.3)$$

We begin by parameterizing sets of the form $\{x \in R^n \mid L_{g_u} V(x) = \psi^T\}$ for $\psi \in R^m$. For a system of the form (1.1) and a Lyapunov function (2.2), we have

$$L_{g_u} V(x) = 2x_N^T[P_{NN} g_u^N(x_N) + P_{NL} g_u^L(x_N)] + 2x_L^T Y(x_N)$$

$$Y(x_N) \doteq P_{LN} g_u^N(x_N) + P_{LL} g_u^L(x_N)$$

To simplify the algebra, we make the following assumption.

Assumption 2.3

For all $x_N \in R^k$, rank $Y(x_N) = m \leq n - k$.

Assumption 2.3 can be relaxed, if necessary, with some modifications to the analysis that follows. Theorem 2.4 shows how the set $\{x \in R^n \mid L_{g_u} V(x) = \psi^T\}$ can be parameterized by x_N and a parameter $\lambda \in R^{n-k-m}$.

Theorem 2.4

Given a function V(x) of the form (2.2), and a vector $\psi \in R^m$,

$$\{x \in R^n \mid L_{g_u} V(x) = \psi^T\} = \left\{\begin{bmatrix} x_N \\ G(x_N)\lambda - P_{LL}^{-1} P_{LN} x_N - \xi(x_N, \psi) \end{bmatrix}, x_N \in R^k, \lambda \in R^{n-k-m}\right\}$$

$$\xi(x_N, \psi) \doteq P_{LL}^{-1} Y(x_N)\, [Y(x_N)^T P_{LL}^{-1} Y(x_N)]^{-1} [g_u^N(x_N)^T R x_N - \frac{1}{2}\psi]$$

$$R \doteq P_{NN} - P_{NL} P_{LL}^{-1} P_{LN}$$

for any matrix $G(x_N) \in R^{(n-k)\times(n-k-m)}$ of full rank such that $Y(x_N)^T G(x_N) \equiv 0$.

Proof

Note that since $G(x_N)$ is full rank, $G(x_N)\lambda$ completely characterizes the null space of $Y(x_N)^T$. Therefore, if any element in the set described above is an element of $\{x \in R^n \mid L_{g_u}V(x) = \psi^T\}$, then that set is equal to $\{x \in R^n \mid L_{g_u}V(x) = \psi^T\}$. Hence, it is sufficient to check the value of $L_{g_u}V(x)$ when $\lambda = 0$.

$$\begin{aligned} L_{g_u}V(x) &= 2x_N^T[P_{NN}g_u^N(x_N) + P_{NL}g_u^L(x_N)] + 2x_L^T Y(x_N) \\ &= 2x_N^T[P_{NN}g_u^N(x_N) + P_{NL}g_u^L(x_N)] - 2[P_{LL}^{-1}P_{LN}x_N + \\ &\qquad \xi(x_N,\psi)]^T Y(x_N) \\ &= 2x_N^T R g_u^N(x_N) - 2[g_u^N(x_N)Rx_N - \frac{1}{2}\psi]^T = \psi^T \end{aligned}$$

With the parameterization of Theorem 2.4, we can express $V(x)$ restricted to $\{x \in R^n \mid L_{g_u}V(x) = \psi^T\}$ as follows

$$V(x_N, \psi, \lambda) = x_N^T R x_N + \xi(x_N,\psi)^T P_{LL}\xi(x_N,\psi) + \lambda^T G(x_N)^T P_{LL} G(x_N)\lambda$$

In order to grid $\mathcal{Y}(c_2)$, we need bounds on the values (x_N, ψ) can take on this set. The permissible values of x_N are those satisfying $x_N^T R x_N \le c_2$. Now for fixed x_N, $L_{g_u}V(x)$ is affine in x_L for systems of the form (1.1). Hence, each element of $L_{g_u}V(x)$ can be represented as $[L_{g_u}V(x)]_p = a_p(x_N) + x_L^T b_p(x_N)$, where

$$\begin{aligned} [a_1(x_N) \cdots a_m(x_N)] &= 2x_N^T[P_{NN}g_u^N(x_N) + P_{NL}g_u^L(x_N)] \\ [b_1(x_N) \cdots b_m(x_N)] &= 2Y(x_N) \end{aligned}$$

The bounds on each element ψ_p are therefore given as follows.

$$\min_{V(x)\le c_2} a_p(x_N) + x_L^T b_p(x_N) \le \psi_p \le \max_{V(x)\le c_2} a_p(x_N) + x_L^T b_p(x_N)$$

By the method of Lagrange multipliers (Ref. [29]), we find this to be equivalent to

(2.4) $\alpha_p(x_N) - \beta_p(x_N) \le \psi_p \le \alpha_p(x_N) + \beta_p(x_N)$

where, for $x_N^T R x_N \le c_2$,

$$\begin{aligned} \alpha_p(x_N) &\doteq a_p(x_N) - b_p(x_N)^T P_{LL}^{-1}P_{LN}x_N \\ \beta_p(x_N) &\doteq \sqrt{[c_2 - x_N^T R x_N][b_p(x_N)^T P_{LL}^{-1} b_p(x_N)}) \end{aligned}$$

The bounds in (2.4) give us the set of values of $L_{g_u}V(x)$ over which we must grid in order to complete the stability analysis. If $\psi = 0$ is within the allowable range for a given x_N, we should use this as one of the grid points to ensure that at least the condition for stability from Remark 1.7 for the case of unlimited control ($\mathcal{U} = R^m$) is satisfied.

By Theorem 2.4, we can parameterize the stability condition as follows for each $(x_N, \psi) \in \mathcal{Y}(c_2)$

$$\Gamma(x_N,\psi,\lambda) = \max_{w \in \mathcal{W}} a_0(x_N,\psi) + b_0(x_N,\psi)^T\lambda + \lambda^T C_0(x_N)\lambda + w^T[s(x_N,\psi) + T(x_N)\lambda] \quad (2.5)$$

The coefficients can be found by straightforward algebra. We can check the condition $\Gamma(c_1,c_2,x_N,\psi) \le 0$ by solving an LMI feasibility problem using the $\mathcal{S}$-procedure as discussed in Ref. [6]. Let us first consider the case of no disturbances ($\mathcal{W} = \{0\}$), for which the parameterized stability condition (2.5) gives us

$$\Gamma(c_1,c_2,x_N,\psi) = \max_{c_1 \le V(x_N,\psi,\lambda) \le c_2} a_0(x_N,\psi) + b_0(x_N,\psi)^T\lambda + \lambda^T C_0(x_N)\lambda \quad (2.6)$$

Checking whether $\Gamma(c_1,c_2,x_N,\psi) \le 0$ is therefore a quadratic feasibility problem with quadratic constraints, which can be solved using the $\mathcal{S}$-procedure in the manner discussed in Ref. [6]. With only one quadratic constraint, the $\mathcal{S}$-procedure is nonconservative (Ref. [6]), but a potential problem arises in our case because there are two constraints. Fortunately, the two constraints are never simultaneously active, as shown in the following theorem.

Theorem 2.5

Suppose $(x_N, \psi) \in \mathcal{Y}(c_2)$ and consider the maximization problem (2.6). If $C_0(x_N) \nless 0$, then $\Gamma(c_1,c_2,x_N,\psi) = \Gamma(0,c_2,x_N,\psi)$. If $C_0(x_N) < 0$, one of the following applies for $\lambda^* = -{}^1/_2\, C_0(x_N)^{-1}b_0(x_N,\psi)$.

1. $V(x_N,\psi,\lambda^*) < c_1$, then $\Gamma(c_1,c_2,x_N,\psi) = \Gamma(c_1,\infty,x_N,\psi)$.
2. $V(x_N,\psi,\lambda^*) > c_2$, then $\Gamma(c_1,c_2,x_N,\psi) = \Gamma(0,c_2,x_N,\psi)$.
3. Otherwise, $\Gamma(c_1,c_2,x_N,\psi) = a_0(x_N,\psi) - {}^1/_4\, b_0(x_N,\psi)^T C_0(x_N)^{-1}b_0(x_N,\psi)$.

Proof

See Appendix A.

Consider next the case where $\mathcal{W}$ is polytopic; for example, $\mathcal{W} = \{w \mid \|w\|_\infty \le 1\}$ belongs to this category. Since the expression in (2.5) is affine in w, robust stability can be analyzed exactly using Theorem 2.5 with each of the extreme points of $\mathcal{W}$ substituted for w. This approach gives us the value of

$$\Gamma(c_1,c_2,x_N,\psi) = \max_{w \in \mathcal{V}} \max_{c_1 \le V(x_N,\psi,\lambda) \le c_2} [a_0(x_N,\psi) + s(x_N,\psi)^T w] + [b_0(x_N,\psi) + T(x_N)^T w]^T\lambda + \lambda^T C_0(x_N)\lambda$$

where $\mathcal{V}$ is the set of extreme points of $\mathcal{W}$. Other classes of disturbance constraints can be handled in the $\mathcal{S}$-procedure framework, but we do not enumerate them here.

We now analyze the computational complexity of the stability analysis procedure outlined in this section. To verify whether a desired stability margin is achieved over a given level set, we must parameterize the set $\{x \in R^n \mid L_{g_u}V(x) = \psi^T\}$ and the level set. We then evaluate robust stability over the resulting parameterized set at various grid point values of x_N (and ψ if there are control limitations). Approximate computation times determined numerically for a system influenced by disturbances contained in $\mathcal{W} = \{w \in R^l \mid \|w\|_\infty \le 1\}$ are listed in Table 2.1. The quantity N_c is the number of grid points in $\mathcal{Y}(c_2)$ over each dimension, so that the total number of grid points is roughly N_c^k (or N_c^{k+m} if there are control limitations).

The complexity of this procedure should be compared with the complexity of evaluating Lyapunov derivatives over a level set to determine the region of stability for a general nonlinear system. This problem is discussed in Refs. [8], [9], [10], [14], [15], [21], [25], and [27]. For an arbitrary nonlinear system, it is necessary to grid the level set over all dimensions, so the computation times required to solve this problem to some desired accuracy grow exponentially with the state dimension. In the procedure developed here for the system (1.1), on the other hand, we only grid $(x_N, \psi) \in \mathcal{Y}(c_2)$ and evaluate whether $\Gamma(c_1,c_2,x_N,\psi) \le 0$ at each grid point. By Theorem 2.5, this reduces to an LMI feasibility problem, and there are standard methods for solving this problem to a desired level of accuracy with a computation time that is polynomial in the state dimension (Refs. [6], [24]). For example, the author implemented an ellipsoid algorithm (Ref. [6]) and found the computation time to vary roughly as $(n - k - m)^3$. Due to gridding over $\mathcal{Y}(c_2)$, the computation time is still exponential in $\dim(x_N)$, or in $\dim(x_N) + \dim(u)$ when control constraints are active. In other words, the computational complexity is exponential only in the degree of nonlinearity in the problem.

2.2 Quadratic RCLF with State-Dependent P Matrix

In this section, we fill in the details of the stability analysis procedure for the special case of an RCLF and stability margin that are arbitrarily nonlinear in x_N but quadratic in x_L. In other words, V(x) and W(x) have the form

(2.7) $$V(x) = x^T P(x_N) x$$

(2.8) $$W(x) = x^T Q(x_N) x$$

where $P \in C^1(R^k \to R^{n\times n})$ and $Q \in C^0(R^k \to R^{n\times n})$. We assume that V(x) and W(x) are positive-definite and that the matrices $P(x_N)$ and $Q(x_N)$ are partitioned in the same manner as for the P and Q matrices in Section 2.1.

We begin by parameterizing sets of the form $\{x \in R^n \mid L_{g_u}V(x) = \psi^T\}$ for $\psi \in R^m$. We can simplify the treatment considerably by making the following assumption on the system (1.1).

Assumption 2.6

For all $x_N \in R^k$, $g_u^N(x_N) = 0$.

For a system of the form (1.1) satisfying Assumption (2.6) and a Lyapunov function of the form (2.7), we have

$$L_{g_u} V(x) = 2x_N^T P_{NL}(x_N) g_u^L(x_N) + x_L^T Y(x_N)$$

$$Y(x_N) \doteq 2P_{LL}(x_N) g_u^L(x_N)$$

To simplify the algebra, we make the following assumption.

Assumption 2.7

For all $x_N \in R^k$, rank $g_u^L(x_N) = m \le n - k$.

Note that Assumption 2.7 is equivalent to the condition that rank $Y(x_N)$ for all $x_N \in R^k$. This assumption can be relaxed, if necessary, with some modifications to the analysis that follows. Theorem 2.8 shows how the set $\{x \in R^n \mid L_{g_u}V(x) = \psi^T\}$ can be parameterized by x_N and a parameter $\lambda \in R^{n-k-m}$.

Theorem 2.8

Given a function V(x) of the form (2.7), and a vector $\psi \in R^m$,

$$\{x \in R^n \mid L_{g_u} V(x) = \psi^T\} = \left\{ \begin{bmatrix} x_N \\ \sigma(x_N,\psi) \end{bmatrix}, x_N \in R^k, \lambda \in R^{n-k-m} \right\}$$

$$\sigma(x_N, \psi) \doteq G(x_N)\lambda - P_{LL}(x_N)^{-1} P_{LN}(x_N) x_N + P_{LL}(x_N)^{-1} Y(x_N)[Y(x_N)^T P_{LL}(x_N)^{-1} Y(x_N)]^{-1}\psi$$

Table 2.1

Computation Times for Stability Analysis using a Quadratic RCLF with a Constant P Matrix

Operation	Time (unbounded control)	Time (bounded control)
Solve for $L_{g_u} V(x) = \psi^T$	$\mathcal{O}(N_c^k n^3)$	$\mathcal{O}(N_c^{k+m} n^3)$
Compute $V(x_N,\psi,\lambda)$	$\mathcal{O}(N_c^k n^4)$	$\mathcal{O}(N_c^{k+m} n^4)$
Check $\Gamma(c_1,c_2,x_N,\psi) \le 0$	$\mathcal{O}(2^l N_c^k n^3)$	$\mathcal{O}(2^l N_c^{k+m} n^3)$

for any matrix $G(x_N) \in R^{(n-k)\times(n-k-m)}$ of full rank such that $Y(x_N)^T G(x_N) \equiv 0$.

Proof

The proof of Theorem 2.8 follows the same arguments as the proof of Theorem 2.4.

With the parameterization of Theorem 2.8, we can express V(x) restricted to $\{x \in R^n \mid L_{g_u}V(x) = \psi^T\}$ as follows.

$$V(x_N, \psi, \lambda) = x_n^T R(x_N)x_N + \psi^T[Y(x_N)^T P_{LL}(x_N)^{-1}Y(x_N)]^{-1}\psi + \lambda^T G(x_N)^T P_{LL}(x_N)G(x_N)\lambda$$

$$R(x_N) \doteq P_{NN}(x_N) - P_{NL}(x_N)P_{LL}(x_N)^{-1}P_{LN}(x_N)$$

In order to grid $\mathcal{Y}(c_2)$, we need bounds on the values (x_N, ψ) can take on this set. The permissible values of x_N are those satisfying $x_N{}^T R(x_N)\, x_N \le c_2$. Similarly, the permissible values of $L_{g_u}V(x) = \psi^T$ corresponding to each such value of x_N are those contained in the ellipsoid described by

$$\psi^T[Y(x_N)^T P_{LL}(x_N)^{-1}Y(x_N)]^{-1}\psi \le c_2 - x_N^T R(x_N)x_N$$

This is the set of values of $L_{g_u}V(x)$ over which we must grid in order to complete the stability analysis. We should always use $\psi = 0$ as one of the grid points to ensure that at least the condition for stability from Remark 1.7 for the case of unlimited control ($\mathcal{U} = R^m$) is satisfied.

By Theorem 2.8, we can parameterize the stability condition as follows for each $(x_N, \psi) \in \mathcal{Y}(c_2)$.

$$\Gamma(x_N,\psi,\lambda) = \sup_{w \in \mathcal{W}} a_0(x_N,\psi) + b_0(x_N,\psi)^T\lambda + \lambda^T C_0(x_N,\psi)\lambda + w^T[s(x_N,\psi) + T(x_N,\psi)\lambda] + \sum_{i=1}^{k}[\lambda^T D_i(x_N)\lambda][h_i(x_N)^T\lambda + v_i(x_N)^T w] \quad (2.9)$$

The coefficients can be found by straightforward algebra. We can rearrange the expression for $\Gamma(c_1,c_2,x_N,\psi)$ when $(x_N, \psi) \in \mathcal{Y}(c_2)$ by introducing the following scaling matrix.

$$K(x_N,\psi) \doteq \sqrt{c_2 - V(x_N,\psi,0)}\,[G(x_N)^T P_{LL}(x_N)G(x_N)]^{-1/2}$$

where $M^{-1/2}$ is the positive-definite square root of M^{-1} for $M = M^T > 0$. Then we can replace λ by $K(x_N, \psi)\lambda$ and set $b_0 \leftarrow Kb_0$, $C_0 \leftarrow KC_0K$, $T \leftarrow TK$, $D_i \leftarrow KD_iK$ and $h_i \leftarrow Kh_i$ to obtain

$$\Gamma(c_1,c_2,x_N,\psi) = \max_{\beta(x_N,\psi)\le\lambda^T\lambda\le 1}\ \sup_{w\in\mathcal{W}} a_0(x_N,\psi) + b_0(x_N,\psi)^T\lambda + \lambda^T C_0(x_N,\psi)\lambda + w^T[s(x_N,\psi) + T(x_N,\psi)\lambda] + \sum_{i=1}^{k}[\lambda^T D_i(x_N,\psi)\lambda][h_i(x_N,\psi)^T\lambda + v_i(x_N)^T w] \quad (2.10)$$

where $\beta(x_N, \psi) = [c_1 - V(x_N, \psi, 0)]/[c_2 - V(x_N, \psi, 0)] \le 1$. This alternative form simplifies the development of a method to evaluate the stability condition.

Checking the condition $\Gamma(c_1,c_2,x_N, \psi) \le 0$ is a nonconvex constrained feasibility problem, which is known to be NP-hard (Refs. [7], [23]). However, in practice, good upper and lower bounds can be computed in polynomial time by transforming the problem to a real μ analysis problem of the type developed in Ref. [32]. Let us first consider the case of no disturbances ($\mathcal{W} = \{0\}$). We approximate the parameterized stability condition (2.10) by

$$\tilde{\Gamma}(0,c_2,x_N,\psi) \doteq \max_{\|\lambda\|_\infty \le 1} \phi_0(\lambda)$$

$$\phi_0(\lambda) \doteq a_0 + b_0^T\lambda + \lambda^T C_0\lambda + \sum_{i=1}^{k}\lambda^T h_i(\lambda^T D_i\lambda) \quad (2.11)$$

where we have suppressed the dependence of the coefficients in (2.11) on (x_N, ψ). By assuming $\mathcal{W} = \{0\}$, we guarantee that $c_1 = 0$. This implies that $\beta(x_N, \psi) \le 0$; in other words, the constraint set in (2.10) is given by $\|\lambda\|_2 \le 1$. It is useful to the development of the procedure that follows that this set is convex. We also replace the condition $\|\lambda\|_2 \le 1$ by $\|\lambda\|_\infty \le 1$. Since $\|\lambda\|_\infty \le \|\lambda\|_2$, this approach is conservative because $\Gamma(0,c_2,x_N,\psi) \le \tilde{\Gamma}(0,c_2,x_N,\psi)$. Nevertheless, we obtain the following test for stability, which is adapted from Proposition 2.1.

Proposition 2.9

If $\mathcal{W} = \{0\}$, V(x) is an RCLF with stability margin W(x) with controls in $\mathcal{U}$ over $V^{-1}[0,c_2]$ if $\tilde{\Gamma}(0,c_2,x_N,\psi) \le 0$ for all $(x_N, \psi) \in \mathcal{Y}(c_2)$.

We now show how Proposition 2.9 can be converted to a real μ analysis problem. Recall that in the real μ analysis problem, we are given a matrix M and an uncertainty class $\boldsymbol{\Delta}$ and we seek to evaluate

$$\mu(M,\boldsymbol{\Delta}) \doteq \begin{cases} \left[\min_{\Delta\in\boldsymbol{\Delta}}\{\bar{\sigma}(\Delta) \mid \det(I+M\Delta)=0\}\right]^{-1} & \exists\,\Delta\in\boldsymbol{\Delta} \mid \det(I+M\Delta)=0. \\ 0 & \text{otherwise} \end{cases}$$

In other words, $\mu(M, \boldsymbol{\Delta})$ is the maximum singular value of the smallest perturbation $\Delta \in \boldsymbol{\Delta}$ that destabilizes a linear fractional interconnection between M and Δ by the small gain theorem (Ref. [32]). Our objective here is to transform Proposition 2.9 to a problem of computing $\mu(M, \boldsymbol{\Delta})$ for some relevant M and $\boldsymbol{\Delta}$.

The following theorem is a general result on how to convert the problem of maximizing any rational function of a constrained variable to a real μ analysis problem.

Theorem 2.10 [Ref. [7]]

For any $q \ge 0$ and any expression $\phi_0(\lambda)$ that can be expressed as a linear fractional transformation of λ and λ^T, there exists a matrix M and an uncertainty class Δ such that

$$\max_{\|\lambda\|_\infty \le 1} \left|\phi_0(\lambda)\right| < q \Leftrightarrow \mu(M,\Delta) < q \qquad (2.12)$$

A basic idea of how to compute M and Δ is given in Ref. [7], where this is done (in slightly more generality) for a quadratic $\phi_0(\lambda)$. We seek to apply Theorem 2.10 when $\phi_0(\lambda)$ has the form (2.11), where $C_0 = C_0^T$ and $D_i = D_i^T$ for all i = 1,..., k. Following Ref. [7], the first step is to generate a block diagram representation for $\phi_0(\lambda)$ from (2.11) such that $y = \phi_0(\lambda)d$ for $d, y \in R$. To this end, we define $e \doteq [1, ..., 1]^T$ with $\dim(e) = \dim(\lambda)$ and $\Lambda \doteq \text{diag}(\lambda)$, so that $\lambda = \Lambda e$. Then $\phi_0(\lambda)$ is represented in block diagram form as in Figure 2.1.

Note that the Λ block is repeated k + 2 times in Figure 2.1. It is useful to transform the block diagram to the alternative form shown in Figure 2.2, where

$$N = \left[\begin{array}{ccc|cc|c} 0 & \cdots & 0 & D_1 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & D_k & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & e \\ h_1e^T & \cdots & h_ke^T & C_0 & 0 & 0 \\ 0 & \cdots & 0 & b_0^T & e^T & a_0 \end{array}\right]$$

$$\Delta = \left[\begin{array}{ccc|cc|c} \Lambda & & & & & \\ & \ddots & & & & \\ & & \Lambda & & & \\ \hline & & & \Lambda & & \\ & & & & \Lambda & \\ \hline & & & & & \delta \end{array}\right]$$

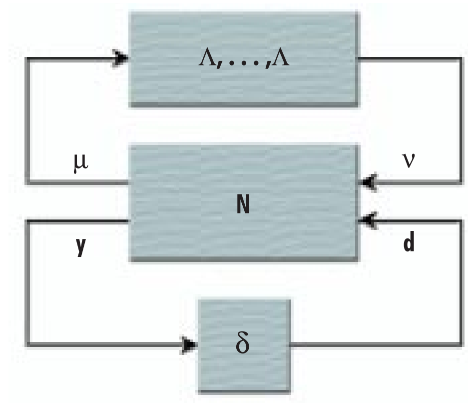

Figure 2.2

Block diagram from Figure 2.1 with uncertainty representation

Note from (2.12) and the discussion in Ref. [7] that the condition $|\phi_0(\lambda)| < q$ is equivalent to $\mu(M, \Delta) < q$ for any $q > 0$, where M is obtained by multiplying all but the last row of N by q.

At this stage we notice that $|\phi_0(\lambda)|$ is used in Theorem 2.10, but what we are really interested in is the maximum of $\phi_0(\lambda)$ itself over $\|\lambda\|_\infty \le 1$. Fortunately, this distinction is not restrictive. Indeed, it is straightforward to find a crude lower bound L such that $\phi_0(\lambda) \ge L$ for all λ satisfying $\|\lambda\|_\infty \le 1$. Given such a lower bound, we can define $\tilde{\phi}_0(\lambda) \doteq \phi_0(\lambda) - L$, and we have $|\tilde{\phi}_0(\lambda)| = \tilde{\phi}_0(\lambda)$ whenever $\|\lambda\|_\infty \le 1$. By constructing $\tilde{M}$ based on $\tilde{\phi}_0(\lambda)$ rather than $\phi_0(\lambda)$, we obtain the following necessary and sufficient condition for a given bound on $\tilde{\Gamma}(0, c_2, x_N, \psi)$ to hold.

$$\mu(\tilde{M}, \Delta) < q \Leftrightarrow \tilde{\Gamma}(0, c_2, x_N, \psi) < q + L$$

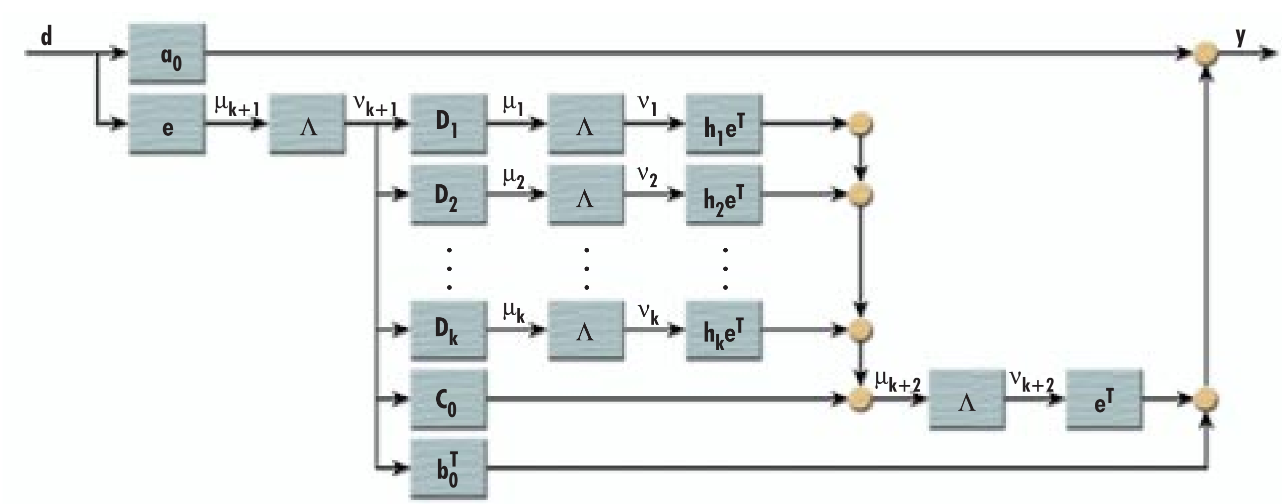

Figure 2.1

Block diagram representation of $\phi_0(\lambda)$ from (2.11)

In other words, the condition in Proposition 2.9 holds if $\mu(\tilde{M}, \Delta) < -L$. Therefore, we can evaluate the stability condition using the standard real μ analysis procedure from Ref. [33].

Consider next the case where $\mathcal{W} = \{w \in R^l \mid \|w\|_\infty \leq 1\}$. Since the expression in (2.10) is affine in w, robust stability can be analyzed exactly by solving the nonconvex feasibility problem with each of the extreme points of $\mathcal{W}$ substituted for w. Alternatively, we could add a block $W \doteq \text{diag}(w)$ to Figures 2.1 and 2.2 so that $y = \phi_0(\lambda)d$, where

$$\tilde{\Gamma}(0, c_2, x_N, \psi) \doteq \max_{\|\lambda\|_\infty \leq 1} \max_{\|w\|_\infty \leq 1} \phi_0(\lambda)$$

$$\phi_0(\lambda) \doteq a_0 + b_0^T\lambda + \lambda^T C_0 \lambda + w^T[s + T\lambda] + \sum_{i=1}^{k} (\lambda^T D_i \lambda)(h_i^T \lambda + v_i^T w)$$

Note, however, that we can only analyze stability in the special case $c_1 = 0$ using this procedure because the level set must be convex. In general, we expect the stability condition to be violated over the level set $V^{-1}[0,c_2]$ unless a matching condition on $g_w(x_N)$ and $g_u(x_N)$ holds.

We now analyze the computational complexity of the stability analysis procedure outlined in this section. The parameterizations of the set $\{x \in R^n \mid L_{g_u}V(x) = \psi^T\}$ and of the level set do not differ significantly from those used in Section 2.1. Robust stability analysis can be accomplished either by general techniques for solving nonconvex constrained feasibility problems or by conversion to a real μ analysis problem. The nonconvex solution procedure is NP-hard (Refs. [7], [23]), but the solution (if it can be found) is exact. Alternatively, approximate bounds on $\tilde{\Gamma}(0,c_2,x_N,\psi)$ can be found using standard techniques for real μ analysis. The computational complexity of this approximation is analyzed in Ref. [33]. Approximate computation times for a system influenced by disturbances contained in $\mathcal{W} = \{w \in R^l \mid \|w\|_\infty \leq 1\}$ are listed in Table 2.2. The quantity N_c is the number of grid points in $y(c_2)$ over each dimension, so that the total number of grid points is roughly N_c^k (or N_c^{k+m} if there are control limitations). The quantity $N_\Delta \doteq (n - k - m)(k + 2) + l + 1$ is the dimension of the perturbation block Δ. The nonconvex solution procedure is at least as complex as gridding over the entire state space to evaluate the Lyapunov derivative.

3. OPTIMIZATION OVER THE RCLF

Now we turn our attention to the problem of constructing an RCLF such that the volume of the level set for guaranteed stability is maximized. This problem formulation is inspired by Ref. [10], in which a quadratic Lyapunov function is used to find the stability region of maximum volume for an arbitrary nonlinear autonomous system of the form $\dot{x} = f(x)$. For this problem, we do not wish to impose a particular stability margin, so we consider the following modified definition for a function to be an RCLF.

Definition 3.1

Consider a subset $\mathcal{W} \subseteq R^l$, a closed subset $\mathcal{U} \subseteq R^m$, and real numbers $c_2 > c_1 > 0$. A proper, positive-definite C^1 function $V(x)$ is a robust control Lyapunov function (RCLF) with controls in $\mathcal{U}$ over $V^{-1}[c_1,c_2]$ for the system (1.2) if there exists a control law $\mu : R^n \to \mathcal{U}$ such that

$$\sup_{x \in V^{-1}[c_1,c_2]} \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w + L_{g_u} V(x) \mu(x) < 0$$

Equivalently,

$$\sup_{x \in V^{-1}[c_1,c_2]} \inf_{u \in \mathcal{U}} \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w + L_{g_u} V(x) u < 0$$

Remark 3.2

If $\mathcal{U} = R^m$, the condition in Definition 3.1 is equivalent to

$$\sup_{x \in V^{-1}[c_1,c_2] \cap \ker(L_{g_u} V)} \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w < 0$$

Note that the analysis of stability over a level set in this scenario is slightly different from the procedure used in Section 2 because we require the Lyapunov derivative to be strictly negative. In other words, if we define

Table 2.2

Computation Times for Stability Analysis Using a Quadratic RCLF with a State-Dependent P Matrix

Operation	Time (unbounded control)	Time (bounded control)
Solve for $L_{g_u}V(x) = \psi^T$	$\mathcal{O}(N_c^k n^3)$	$\mathcal{O}(N_c^{k+m} n^3)$
Compute $V(x_N,\psi,\lambda)$	$\mathcal{O}(N_c^k n^4)$	$\mathcal{O}(N_c^{k+m} n^4)$
Nonconvex solution	$\mathcal{O}(2^l N_c^n)$	$\mathcal{O}(2^l N_c^n)$
μ upper bound	$\mathcal{O}(N_c^k N_\Delta^3)$	$\mathcal{O}(N_c^{k+m} N_\Delta^3)$

$$\Gamma(x_N,\psi,\lambda) \doteq \sup_{w \in \mathcal{W}} L_f V(x) + L_{g_w} V(x) w + \left[\inf_{u \in \mathcal{U}} \psi^T u\right]$$

$$\Gamma(c_1,c_2,x_N,\psi) \doteq \max_{\lambda \in \mathcal{L}(c_1,c_2,x_N,\psi)} \Gamma(x_N,\psi,\lambda)$$

then a condition for V(x) to be an RCLF is given by Proposition 3.3.

Proposition 3.3

V(x) is an RCLF with controls in $\mathcal{U}$ over $V^{-1}[c_1,c_2]$ if and only if $\Gamma(c_1,c_2,x_N,\psi) < 0$ for all $(x_N,\psi) \in \mathcal{Y}(c_2)$.

Since we are optimizing over V(x) at this point, we do not also need to iterate over the level values to determine the largest region of stability. Therefore, we shall henceforth consider the problem of finding the function V(x), which is an RCLF with controls in $\mathcal{U}$ over $V^{-1}[\gamma, 1]$ for some $\gamma > 0$ such that the volume of the set $V^{-1}[0, 1]$ is maximized. The appropriate value of γ depends on the RCLF. However, if we assume that there exists some nominal $V_0(x)$, which is an RCLF with controls in $\mathcal{U}$ over $V_0^{-1}[c_1,c_2]$, then we could fix γ to be the largest value satisfying $V^{-1}[0,\gamma] \subseteq V_0^{-1}[0,c_2]$. In this way, we can find a control law that renders the system RUAS($\mathcal{X}$, Ω) with $\mathcal{X} = V^{-1}[0, 1]$ and Ω over $V_0^{-1}[0,c_1]$ by the method in Ref. [19]. In other words, the inner level set over which stability is guaranteed is independent of the particular RCLF used. Therefore, a direct comparison of the volumes of sets of the form $V^{-1}[0, 1]$ for different RCLFs is a meaningful comparison of the size of the stability region.

In Sections 3.1-3.2, we describe the optimization procedures that might be applied to the two Lyapunov function classes of Section 2 and evaluate the computational complexity of the procedure in each case.

3.1 Quadratic RCLF with Constant P Matrix

Our objective is to find a quadratic function V(x) of the form (2.2), which is an RCLF with controls in $\mathcal{U}$ over $V^{-1}[\gamma, 1]$ and which maximizes the volume of the level set $V^{-1}[0, 1]$. Following Ref. [10], we can pose the following optimization problem.

Problem 3.4

Suppose that $V_0(x) \doteq x^T P_0 x$ is an RCLF with controls in $\mathcal{U}$ over $V^{-1}[c_1,c_2]$. Determine $P = P^T$ and γ from the following optimization problem.

$$\begin{aligned} &\text{minimize} && \det(P) \\ &\text{subject} && \gamma > 0 \\ &&& P/\gamma \geq P_0 / c_2 \\ &&& \Gamma(\gamma,1,x_N,\psi) < 0,\ \forall (x_N,\psi) \in \mathcal{Y}(1) \end{aligned}$$

Problem 3.4 is a nonlinear, nonconvex optimization problem in the variables P and γ. From Ref. [6], we note that by casting the problem in terms of P^{-1} instead of P, we can rewrite the objective using the convex function log det(P^{-1}). However, the stability condition $\Gamma(\gamma,1,x_N,\psi) < 0$ is nonconvex in both P and P^{-1}, and there is no obvious way to reformulate Problem 3.4 to be convex. Consequently, Problem 3.4 is NP-hard. In particular, the computational complexity of this problem is comparable to the complexity of gridding over the set of symmetric positive-definite matrices P. Note that the largest value of γ that satisfies the constraint $P/\gamma \geq P_0/c_2$ can be computed exactly by $\gamma = c_2 \lambda_{min}(P P_0^{-1})$.

An approximate solution to Problem 3.4 is developed in Ref. [11]. If the nonlinear terms in the dynamics of (1.1) are rational functions of x_N, then the system can be written in the following linear fractional representation.

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u + B_p p \\ q &= C_q x + D_{qw} w + D_{qu} u + D_{qp} p \\ p &= \Delta(x) q \\ \Delta(x_N) &= \text{diag}(x_1 I_{r_1}, \ldots, x_k I_{r_k}) \end{aligned}$$

In other words, the nonlinear dynamics of the system are written as a linear fractional transformation (LFT) between a nominal linear time-invariant system and a perturbation block containing (possibly repeated) values of the "nonlinear" states x_N. Note that $\dim(q) = \dim(p) = \sum_{i=1}^{k} r_i$, which is the size of the perturbation block required to represent the nonlinear terms in the dynamics. This quantity could possibly be much larger than dim(x_N), depending on the types of nonlinearities present.

The control design method proposed in Ref. [11] could therefore be used to obtain an RCLF $V(x) = x^T Px$. Since the problem formulation in Ref. [11] is an LMI convex optimization problem, we can use this procedure to try to optimize any convex function of the matrix P subject to the constraint of stability over a level set. However, there are two important sources of conservatism in this procedure. The first is that a linear controller structure is assumed, and the solution to the optimization problem under this assumption may not yield the optimal Lyapunov function when nonlinear control laws are allowed. The conservatism due to this assumption could probably be reduced by assuming an LFT structure for the control as well as the plant, in a manner analogous to the incorporation of time-varying parameters in the control for an LPV system in Ref. [26].

The second source of conservatism is probably more important. The stability constraint in Ref. [11] is derived from the small gain theorem from robust control and actually requires that the following system is stable.

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u + B_p p \\ q &= C_q x + D_{qw} w + D_{qu} u + D_{qp} p \\ p &= \Delta(t) q \end{aligned}$$

In this alternative system description, the perturbation block is allowed to be any matrix function $\Delta(t) \in \boldsymbol{\Delta}$ for some constraint set $\boldsymbol{\Delta}$ representing an upper bound on the allowable magnitude of the nonlinear states over the appropriate level set of V(x). Treating nonlinearities as disturbances and applying robust control techniques is known to be an overly conservative approach to nonlinear control (Ref. [1]). Consequently, there is no guarantee that the matrix P obtained using this procedure solves the original optimization problem. Nevertheless, this procedure is an alternative that may yield good results in some cases.

We now analyze the computational complexity of the optimization procedure described in this section. The optimization can be accomplished either by general nonconvex optimization techniques or by solving an equivalent real μ analysis problem applied to the LFT representation. The general nonconvex optimization problem is NP-hard (Ref. [23]), but the solution (if it can be found) is exact. The number of variables in this problem is equal to $n(n + 1)/2$, the dimension of the set of symmetric matrices. Alternatively, approximate bounds on $\Gamma(\gamma,1,x_N,\psi)$ can be found using standard techniques for real μ analysis (Ref. [33]). Approximate computation times for a system influenced by disturbances contained in $\mathcal{W} = \{w \in R^l \mid \|w\|_\infty \leq 1\}$ are listed in Table 3.1. Note that the perturbation block in the robust stability problem with an LFT representation for the nonlinear terms has dimension $l + r$, where $r = \sum_{i=1}^{k} r_i$.

An alternative to the LFT representation would be to regard the system (1.1) as a quasi-LPV system and apply the procedure from Ref. [4] for LPV systems. In this procedure, an RCLF is computed by solving a family of LMI convex optimization problems parameterized over the "nonlinear" states x_N. Here again the computational complexity is exponential in k (exponential in k + m in the bounded control case) and polynomial in the remaining state dimension. Note, however, that LPV systems are slightly different from the system (1.1); in (1.1) the dynamics of x_N are known and x_N is part of the state vector to be regulated to a desired equilibrium point.

3.2 Quadratic RCLF with State-Dependent P Matrix

The problem of optimizing over a function V(x) of the form (2.7) is substantially more complicated than the problem in Section 3.1. The optimization problem under consideration here is the following.

Problem 3.5

Suppose that $V_0(x) \doteq x^T P_0(x_N) x$ is an RCLF with controls in $\mathcal{U}$ over $V_0^{-1}[c_1,c_2]$. Determine $P(x_N) = P^T(x_N)$ and γ from the following optimization problem.

$$\begin{aligned} &\text{minimize} && \text{vol}\, V^{-1}[0,1] \\ &\text{subject to} && \gamma > 0 \\ &&& V^{-1}[0,\gamma] \subseteq V_0^{-1}[0,c_1] \\ &&& \Gamma(\gamma,1,x_N,\psi) < 0,\ \forall (x_N,\psi) \in \mathcal{Y}(1) \end{aligned}$$

To develop a tractable approximate solution to Problem 3.5, we would like to view the problem as an optimization over $P(x_N)$, which is analogous to Problem 3.4 but parameterized by the x_N states. We can write the objective and the second constraint in Problem 3.5 in terms of $P(x_N)$, although the formulas may be complicated. In the stability constraint, however, the derivatives of $P(x_N)$ with the x_N states appear in $\Gamma(\gamma,1,x_N,\psi)$. Therefore, we cannot simply evaluate this constraint at a given value of x_N independent of the others; we really need to optimize over the whole function $P(x_N)$. Consequently, we cannot view Problem 3.5 as a parameterized collection of subproblems for fixed x_N as we would like. We do not currently have a procedure for solving Problem 3.5, even approximately, which is not NP-hard.

4. NUMERICAL EXAMPLE

In the system shown in Figure 4.1, a pole is hinged on a cart, and a spring joins the top of the pole to a fixed point on the wall behind the cart. The control is a force on the cart, which is limited by a saturation constraint. We want a control design to drive the system to the origin from an

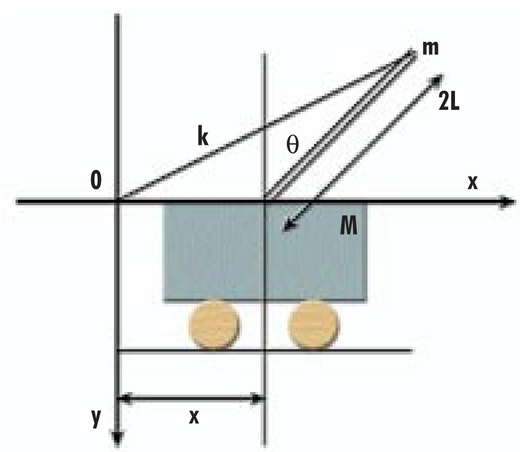

Figure 4.1

Cart with inverted pendulum and spring

Table 3.1

Computation Times for Optimization Using a Quadratic RCLF with a Constant P Matrix

Operation	Time (unbounded control)	Time (bounded control)
Nonconvex solution	$\mathcal{O}(2^l N_c^n)$	$\mathcal{O}(2^l N_c^n)$
LFT solution	$\mathcal{O}(N_c^k [l+r]^3)$	$\mathcal{O}(N_c^{k+m} [l+r]^3)$

initial condition. A simplified model of the dynamics has the form

$$\begin{bmatrix} \dot{\theta} \\ \dot{x}_L \end{bmatrix} = f(\theta) + A(\theta)x_L + g_u(\theta)u$$

where $x_N = \theta$ and $x_L = [\dot{\theta}; x; \dot{x}]$.

After designing an RCLF V(x) and stability margin W(x), we analyze stability over $V^{-1}[0,1]$, under the saturation constraint using the method described in Section 2.1. A comparison of the computational complexity of this procedure with that of a "brute-force" gridding procedure for different grid densities N_c is shown in Figure 4.2. This example shows the computational complexity increasing as a function of N_c with a higher exponent for the case of the brute-force grid than for the method of Section 2.1. Specifically, for large N_c the complexity increases as N_c^4 for the brute-force grid and as N_c^2 for the method of Section 2.1, as expected.

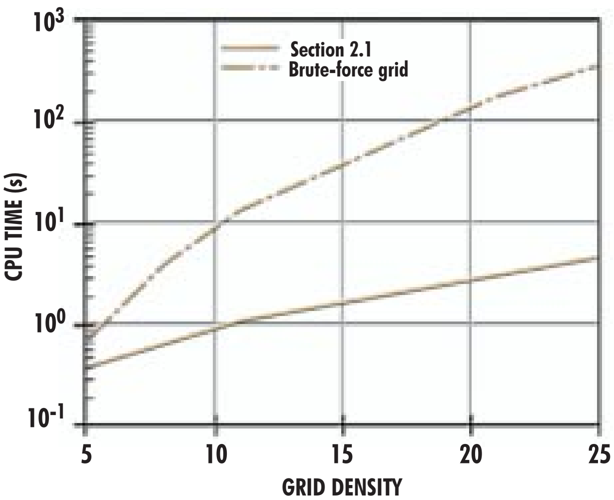

Figure 4.2

Computation time required for analysis with constant P using a Sparc 20 processor

5. CONCLUSIONS

The most serious hindrance to progress on the nonlinear control problem is the inherent complexity of the class of arbitrary nonlinear systems. Therefore, it is important to identify a class of systems that is sufficiently restricted so that computations can be made tractable, yet that is general enough to apply to a wide variety of real control systems. We have shown some ways in which systems of the form (1.1) form such a class. When the dynamics are nonlinear in only a fixed number of variables x_N, the computation time required for stability analysis given a quadratic RCLF is exponential only in $\dim(x_N)$ and polynomial in the dimension of the remaining states. For an RCLF of the form $V(x) = x^T P(x_N)x$, the stability analysis problem is NP-hard. In practice, however, good bounds on the Lyapunov derivative can be found by converting the problem to a real μ analysis problem. This results in an approximate solution procedure with a computational complexity that is exponential in $\dim(x_N)$ and polynomial in the dimension of the remaining states.

Similarly, the problem of optimizing over RCLFs to get the largest possible stability region is NP-hard. We can get a computationally tractable, approximate solution procedure by finding an LFT representation for the system dynamics. The computational complexity of this problem is exponential only in $\dim(x_N)$. The computation time also depends polynomially on the dimension of the perturbation block needed to represent the nonlinear dynamics. Since this procedure relies on bounding the nonlinear terms in the dynamics and applying robust control techniques, we expect this procedure to be overly conservative. We do not obtain a computationally tractable optimization procedure for RCLFs of the form $V(x) = x^T P(x_N)x$.

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APPENDIX A: PROOF OF THEOREM 2.5

Suppose that $C_0(x_N) \nless 0$. It is clear that $\Gamma(c_1,c_2,x_N,\psi) \leq \Gamma(0,c_2,x_N,\psi)$. Let $\lambda^* \in \mathcal{L}(0,c_2,x_N,\psi)$ be such that $\Gamma(x_N,\psi,\lambda^*) = \Gamma(0,c_2,x_N,\psi)$. By the $\mathcal{S}$-procedure, this quantity is equal to the smallest r such that $\tau \geq 0$ exists to satisfy the following for *all* values of λ.

(A.1) $\qquad -\Gamma(x_N,\psi,\lambda) + r + \tau[V(x_N,\psi,\lambda) - c_2] \geq 0$

Therefore, $\tau[V(x_N,\psi,\lambda^*) - c_2] \geq 0$. On the other hand, the constraint and $\tau \geq 0$ combine to give $\tau[V(x_N,\psi,\lambda^*) - c_2] \leq 0$. Therefore, $\tau[V(x_N,\psi,\lambda^*) - c_2] = 0$. Since (A.1) must hold for all λ, we require $C_0(x_N) - \tau C_1(x_N) \leq 0$ for $C_1(x_N) = G(x_N)^T P_{LL} G(x_N) > 0$. Hence, if $C_0(x_N) \nleq 0$, then we require $\tau > 0$, which implies $V(x_N,\psi,\lambda^*) = c_2$. If $C_0(x_N) \leq 0$ but $C_0(x_N) \nless 0$, then there exists $\mu \neq 0$ such that $C_0(x_N)\mu = 0$. We want to show that for any λ satisfying $V(x_N,\psi,\lambda) < c_2$, there exists $\beta \in R$ such that $\Gamma(x_N,\psi,\lambda + \beta\mu) \geq \Gamma(x_N,\psi,\lambda)$ but $V(x_N,\psi,\lambda + \beta\mu) = c_2$, and this will complete the proof. Now $V(x_N,\psi,\lambda + \beta\mu) = [\mu^T C_1(x_N)\mu]\beta^2 + [2\mu^T C_1(x_N)\lambda]\beta + V(x_N,\psi,\lambda)$. Therefore, the equation $V(x_N,\psi,\lambda + \beta\mu) = c_2$ is quadratic in β, and if $V(x_N,\psi,\lambda) < c_2$, there are two real roots: one positive and one negative. Since $\Gamma(x_N,\psi,\lambda + \beta\mu) = \Gamma(x_N,\psi,\lambda) + \beta b_0(x_N,\psi)^T\mu$, and at least one of the roots has the property

$\beta b_0(x_N,\psi)^T\mu \geq 0$, we get $\Gamma(x_N,\psi,\lambda + \beta\mu) \geq \Gamma(x_N,\psi,\lambda)$ and $V(x_N,\psi,\lambda + \beta\mu) = c_2$. Therefore, there exists λ^* with the property $V(x_N,\psi,\lambda^*) = c_2$ and $\Gamma(x_N,\psi,\lambda^*) = \Gamma(0,c_2,x_N,\psi)$, and we conclude that $\Gamma(c_1,c_2,x_N,\psi) = \Gamma(0,c_2,x_N,\psi)$. Finally, suppose that $C_0(x_N) < 0$. Then $\Gamma(x_N,\psi,\lambda)$ is concave and is maximized for $\lambda^* = -{}^1/_2 C_0(x_N)^{-1} b_0(x_N,\psi)$. If $\lambda^* \notin \mathscr{L}(c_1,c_2,x_N,\psi)$, then $\Gamma(x_N,\psi,\lambda)$ is maximized on the boundary of $\mathscr{L}(c_1,c_2,x_N,\psi)$ closest to λ^*. This proves the first two statements. Otherwise, $\Gamma(c_1,c_2,x_N,\psi) = \Gamma(x_N,\psi,\lambda^*) = a_0(x_N,\psi) - {}^1/_4 b_0(x_N,\psi)^T C_0(x_N)^{-1} b_0(x_N,\psi)$, which proves the third statement.

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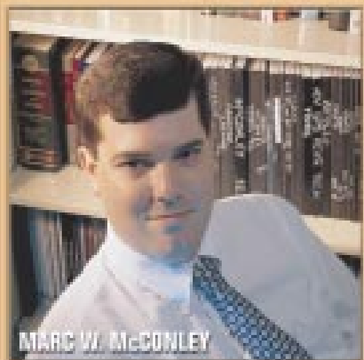
BIOGRAPHIES

Computational Complexity of Lyapunov Stability Analysis Problem for a Class of Nonlinear Systems



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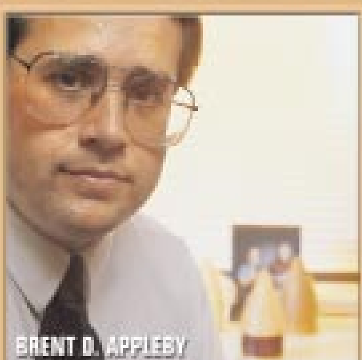

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